The 27^{rd} Balkan Mathematical Olympiad
Tirana, June 25, 2023

Problem 1.

Find all pairs (a, b) of positive integers such that $a! + b$ and $b! + a$ are both powers of 5.

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Solution. The condition is symmetric so we can assume that $b \leq a$.
The first case is when $a = b$. In this case, $a! + a = 5^m$ for some positive integer m. We can rewrite this as $a \cdot ((a-1)! + 1) = 5^m$. This means that $a = 5^k$ for some integer $k \geq 0$. It is clear that k cannot be 0. If $k \geq 2$, then $(a-1)! + 1 = 5^l$ for some $l \geq 1$, but $a - 1 = 5^k - 1 > 5$, so $5|(a-1)!$, which is not possible because $5|(a-1)! + 1$. This means that $k = 1$ and $a = 5$. In this case, $5! + 5 = 125$, which gives us the solution $(5, 5)$.
Let us now assume that $1 \leq b < a$. Let us first assume that $b = 1$. Then $a + 1 = 5^x$ and $a! + 1 = 5^y$ for integers $x, y \geq 1$. If $x \geq 2$, then $a = 5^x - 1 \geq 5^2 - 1 > 5$, so $5|a!$. However, $5|5^y = a! + 1$, which leads to a contradiction. We conclude that $x = 1$ and $a = 4$. From here $a! + b = 25$ and $b! + a = 5$, so we get two more solutions: $(1, 4)$ and $(4, 1)$.
Now we focus on the case $1 < b < a$. Then we have $a! + b = 5^x$ for $x \geq 2$, so $b \cdot \left(\frac{a!}{b} + 1\right) = 5^x$, where $b|a!$ because $a > b$. Because $b|5^x$ and $b > 1$, we have $b = 5^z$ for $z \geq 1$. If $z \geq 2$, then $5 < b < a$, so $5|a!$, which means that $\frac{a!}{b} + 1$ cannot be a power of 5. We conclude that $z = 1$ and $b = 5$. From here $5! + a$ is a power of 5, so $5|a$, but $a > b = 5$, which gives us $a \geq 10$. However, this would mean that $25|a!$, $5|b$ and $25 \nmid b$, which is not possible, because $a! + b = 5^x$ and $25|5^x$.
We conclude that the only solutions are $(1, 4)$, $(4, 1)$ and $(5, 5)$. $\square$

Problem 2.

Prove that for all non-negative real numbers x, y, z, not all equal to 0, the following inequality holds

$$\frac{2x^2 - x + y + z}{x + y^2 + z^2} + \frac{2y^2 + x - y + z}{x^2 + y + z^2} + \frac{2z^2 + x + y - z}{x^2 + y^2 + z} \geqslant 3.$$

Determine all the triples (x, y, z) for which the equality holds.

Solution. Let us first write the expression L on the left hand side in the following way

$$\begin{aligned} L &= \left(\frac{2x^2 - x + y + z}{x + y^2 + z^2} + 2\right) + \left(\frac{2y^2 + x - y + z}{x^2 + y + z^2} + 2\right) + \left(\frac{2z^2 + x + y - z}{x^2 + y^2 + z} + 2\right) - 6 \\ &= (2x^2 + 2y^2 + 2z^2 + x + y + z)\left(\frac{1}{x + y^2 + z^2} + \frac{1}{x^2 + y + z^2} + \frac{1}{x^2 + y^2 + z}\right) - 6. \end{aligned}$$

If we introduce the notation $A = x + y^2 + z^2$, $B = x^2 + y + z^2$, $C = x^2 + y^2 + z$, then the previous relation becomes

$$L = (A + B + C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) - 6.$$

Using the arithmetic-harmonic mean inequality or Cauchy-Schwartz inequality for positive real numbers A, B, C, we easily obtain

$$(A + B + C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geqslant 9,$$

so it holds $L \geqslant 3$.
The equality occurs if and only if $A = B = C$, which is equivalent to the system of equations

$$x^2 - y^2 = x - y, \quad y^2 - z^2 = y - z, \quad x^2 - z^2 = x - z.$$

It follows easily that the only solutions of this system are
$(x, y, z) \in \{(t, t, t) \mid t > 0\} \cup \{(t, t, 1-t) \mid t \in [0, 1]\} \cup \{(t, 1-t, t) \mid t \in [0, 1]\} \cup \{(1-t, t, t) \mid t \in [0, 1]\}$.

□

PSC Remark We feel the equality case needs more explanations in order to have a complete solution, our suggestion follows:

Clearly if x, y, z are all equal and not 0 satisfy the condition. Now suppose that not all of them are equal it means we can't simultaneously have $x + y = y + z = z + x = 1$ otherwise we would have all x, y, z equal to $\frac{1}{2}$ which we already discussed. We can suppose now that $x = y$ and $y + z = z + x = 1$ where we get $z = 1 - x$. So, all triples which satisfy the equality are $(x, y, z) = (a, a, a), (b, b, 1 - b)$ and all permutations for any $a > 0$ and $b \in [0, 1]$

Problem 3.

Alice and Bob play the following game on a 100×100 grid, taking turns, with Alice starting first. Initially the grid is empty. At their turn, they choose an integer from 1 to 100^2 that is not written yet in any of the cells and choose an empty cell, and place it in the chosen cell. When there is no empty cell left, Alice computes the sum of the numbers in each row, and her score is the maximum of these 100 sums. Bob computes the sum of the numbers in each column, and his score is the maximum of these 100 sums. Alice wins if her score is greater than Bob's score, Bob wins if his score is greater than Alice's score, otherwise no one wins.

Find if one of the players has a winning strategy, and if so which player has a winning strategy.

Solution. We denote by (i, j) the cell in the i-th line and in the j-th column for every $1 \leq i, j \leq n$. Bob associates the following pair of cells : $(i, 2k+1), (i, 2k+2)$ for $1 \leq i \leq 100$ and $0 \leq k \leq 49$ except for $(i, k) = (100, 0)$ and $(100, 1)$, and the pairs $(100, 1), (100, 3)$ and $(100, 2), (100, 4)$.
Each time Alice writes the number j in one of the cell, Bob writes the number $100^2 + 1 - j$ in the other cell of the pair.
One can prove by induction that after each of Bob's turn, for each pair of cell, either there is a number written in each of the cell of the pair, or in neither of them. And that if a number j is written, $100^2 + 1 - j$ is also written. Thus Bob can always apply the previous strategy (since $j = 100^2 + 1 - j$ is impossible).
At the end, every line has sum $(100^2 + 1) \times 50$.
Assume by contradiction that Alice can stop Bob from winning if he applies this strategy. Let c_j be the sum of the number in the j-th column for $1 \leq j \leq 100$: then $c_j \leq 50(100^2 + 1)$. Note that :

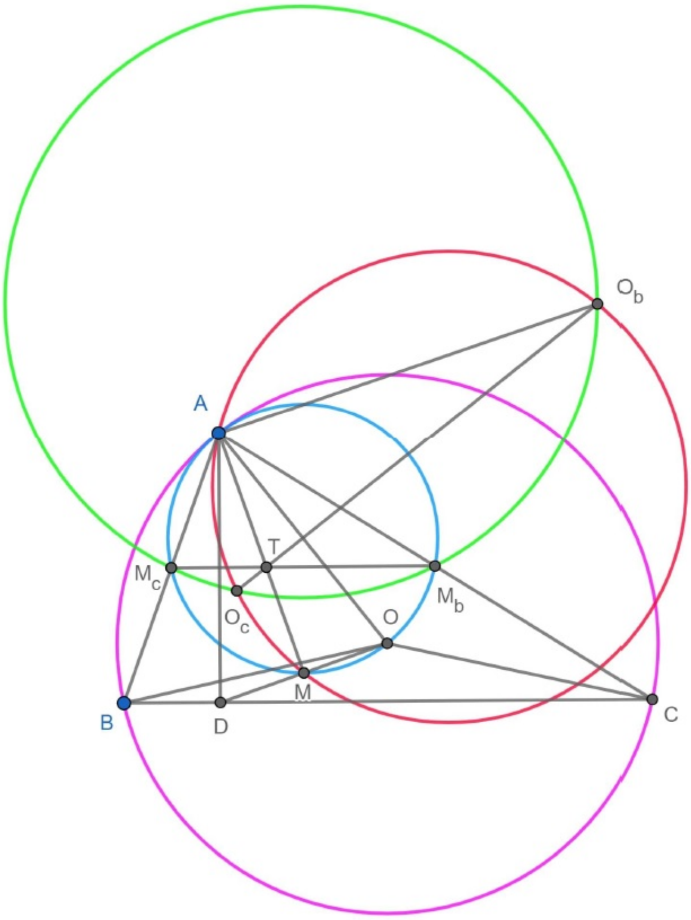
$$100 \times 50(100^2 + 1) \geq c_1 + \cdots + c_{100} = 1 + \cdots + 100^2 = \frac{100^2(100^2 + 1)}{2} = 100 \times 50(100^2 + 1)$$

Thus we have equality in the previous inequality : $c_1 = \cdots = c_{100} = 50(100^2 + 1)$. But if a is the number written in the case $(100, 1)$ and b the number written in the case $(100, 2)$, then $c_1 - b + c_2 - c = 99(100^2 + 1)$. Thus $b + c = 100(100^2 + 1) - 99(100^2 + 1) = 100^2 + 1$: by hypothesis c is also written in the cell $(100, 3)$ which is a contradiction.
Thus Bob has a winning strategy ∎

Problem 4.

Let ABC be an acute triangle with circumcenter O. Let D be the foot of the altitude from A to BC and let M be the midpoint of OD. The points O_b and O_c are the circumcenters of triangles AOC and AOB, respectively. If $AO = AD$, prove that the points A, O_b, M and O_c are concyclic.

Solution.



Note that $AB = AC$ cannot hold since $AO = AD$ would imply that O is the midpoint of BC, which is not possible for an acute triangle. So we may assume without loss of generality that $AB < AC$.

Let M_b and M_c be the midpoints of AC and AB, respectively. Since $\angle AM_bO = \angle AM_cO = 90^\circ = \angle AMO$ (the latter since $AO = AD$), the pentagon AM_bOMM_c is cyclic.

Next, notice that AM is the perpendicular bisector of OD, O_bO_c is the perpendicular bisector of AO and M_bM_c is the perpendicular bisector of AD. Hence these three lines are concurrent – denote their common point by T.

The quadrilateral AO_bMO_C is cyclic if and only if $AT \cdot TM = O_bT \cdot O_cT$. From the cyclic AM_bMM_c we have $AT \cdot TM = M_bT \cdot M_cT$. Hence it now suffices to argue $M_bT \cdot M_cT = O_bT \cdot O_cT$ – or equivalently, that M_b, M_c, O_b and O_c are concyclic.

We assume that $\angle AOB < 90^\circ$ and $\angle AOC > 90^\circ$ so that O_c is in the interior of triangle AOB and O_b in external to the triangles AOC (the other cases are analogous and if $\angle AOB = 90^\circ$ or $\angle AOC = 90^\circ$, then $M_b \equiv O_b$ or $M_c \equiv O_c$ and we are automatically done). We have

$$\angle M_cM_bO_b = 90^\circ + \angle AM_bM_c = 90^\circ + \angle ACB$$

as well as (since O_cO_b is a perpendicular bisector of AO and hence bisects $\angle AO_CO$)

$$\angle M_cO_cO_b = 180^\circ - \angle OO_cO_b = 90^\circ + \frac{\angle AO_cM_c}{2}$$

$$= 90^\circ + \frac{\angle AO_cB}{4} = 90^\circ + \frac{\angle AOB}{2} = 90^\circ + \angle ACB$$

and therefore $O_bM_bO_cM_c$ is cyclic, as desired.

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