JBMO
Albania 2023

Language: **English**

The 27^{th} Junior Balkan Mathematical Olympiad
Tirana, June 25, 2023

Problem 1.

Find all pairs (a, b) of positive integers such that $a! + b$ and $b! + a$ are both powers of 5.

Problem 2.

Prove that for all non-negative real numbers x, y, z, not all equal to 0, the following inequality holds

$$\frac{2x^2 - x + y + z}{x + y^2 + z^2} + \frac{2y^2 + x - y + z}{x^2 + y + z^2} + \frac{2z^2 + x + y - z}{x^2 + y^2 + z} \geqslant 3.$$

Determine all the triples (x, y, z) for which the equality holds.

Problem 3.

Alice and Bob play the following game on a 100×100 grid, taking turns, with Alice starting first. Initially the grid is empty. At their turn, they choose an integer from 1 to 100^2 that is not written yet in any of the cells and choose an empty cell, and place it in the chosen cell. When there is no empty cell left, Alice computes the sum of the numbers in each row, and her score is the maximum of these 100 sums. Bob computes the sum of the numbers in each column, and his score is the maximum of these 100 sums. Alice wins if her score is greater than Bob's score, Bob wins if his score is greater than Alice's score, otherwise no one wins.

Find if one of the players has a winning strategy, and if so which player has a winning strategy.

Problem 4.

Let ABC be an acute triangle with circumcenter O. Let D be the foot of the altitude from A to BC and let M be the midpoint of OD. The points O_b and O_c are the circumcenters of triangles AOC and AOB, respectively. If $AO = AD$, prove that the points A, O_b, M and O_c are concyclic.

Time: 4 hours and 30 minutes

Each problem is worth 10 points